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Limit speed of particles in a non-homogeneous electric field under friction

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Abstract
It is shown that under certain conditions the limit speed of electric charges moving in a space of type $\mathbb{R}^n$ of dimension one or two, under isotropic friction, is preserved under some perturbations. These results hold when relativistic equations of motion are considered.

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1. Introduction

The motion of a charged particle in $\mathbb{R}$ and $\mathbb{R}^2$ under the action of a weakly varying (in space) electric field $\mathbf{E}(\mathbf{x})$ is studied assuming that friction is present. Concretely, the existence or not of a limit speed, which is known to exist when $\mathbf{E}$ is constant, see [1], is considered (see sections 2 and 3). All these results seem to be new.

Results are also obtained (see section 5) for the relativistic equation

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\dot{\mathbf{x}}) = \mathbf{E} - \dot{\mathbf{x}} f(\dot{\mathbf{x}}^2) \\ &\gamma = (1-\dot{\mathbf{x}}^2)^{-1/2} \\ &\mathbf{E} = (0, e(y)) \\ &e(y) \approx e_0,\ e_0 \in \mathbb{R} \\ &\mathbf{x} = (x, y) \\ &\text{mass} = 1 \\ &c = \text{light speed} = 1 \end{aligned}\right\}. \tag{1}$$

The results of this paper are not extendable to electric fields of the form

$$\left.\begin{aligned} &\mathbf{E} = (e_x, e_y) \\ &e_x \approx 0,\ e_y \approx e_0 \end{aligned}\right\}, \tag{2}$$

whose study will be the subject of another paper.

The results of sections 2–5 indicate that under certain conditions on $e(y)$ and $f(\dot{\mathbf{x}}^2)$, the perturbation $(0, e(y))$ of $(0, e_0)$ inherits a limit velocity, even in the relativistic case.

Recent research on the asymptotic behavior of scalar second-order differential equations can be found in [2]. Concerning references with more physical taste, see [3]. Note that in all these papers $\mathbf{E}$ is a constant vector field.

The reader should have in mind the relations between the limit speed of charged particles in an electric field with the experiments of R A Millikan in order to determine the charge of the electron [4]. In these experiments, the frictional force is linear in the velocity [5] its origin being the interaction of charged droplets of oil with a surrounding supersaturated water vapor atmosphere.

This problem arises in the study of the motion of electric charges in the stars and in the accretion disks surrounding certain black holes, where friction between the charges and the surrounding medium is physically important and cannot be neglected. See [1] for a more complete discussion.

2. Limit speed in one dimension

In this section, the limit behavior of $\dot{\mathbf{x}}$ (when $t \to +\infty$) for the differential equations (1), (3), (13) and (15) shall be studied.

Note that in all these cases the limit velocity is a global one; that is, it is valid for all the initial conditions $(\mathbf{x_0}, \dot{\mathbf{x}}_\mathbf{0})$.

We first study the scalar equation

$$\left.\begin{aligned} &\ddot{x} = E_0 + p(x) - \dot{x} \\ &E_0 \in \mathbb{R}^+,\ E_0 + p(x) \geqslant e_0 > 0 \\ &\lim_{x\to+\infty} p(x) = 0 \end{aligned}\right\}, \tag{3}$$

that is, a perturbation of the differential equation

$$\ddot{x} = E_0 - \dot{x}, \tag{4}$$

whose general solution is

$$x(t) = A + Be^{-t} + E_0 t, \tag{5}$$

and therefore

$$\lim_{t\to+\infty} \dot{x}(t) = E_0. \tag{6}$$

We now show that for every solution of equation (3) we also have

$$\lim_{t\to+\infty} \dot{x} = E_0. \tag{7}$$

That is, the limit speed E_0 is preserved for the family of differential equations (3) when the constant electric field is perturbed with the term $p(x)$.

In fact, the vector field (vf in what follows) $\mathbf{X}$ associated with equation (3) is given by

$$\mathbf{X} = (\dot{x}, E_0 + p(x) - \dot{x}), \tag{8}$$

and we have

(A) The orbits of $\mathbf{X}$ are defined for any $t > 0$ since:

- (A.1) $\dot{x}(t)$ is bounded for $t > 0$. Note that $\frac{d\dot{x}}{dt}$ changes its sign on the real curve $\dot{x} = E_0 + p(x)$ in the plane $(x, \dot{x})$.
- (A.2) $x(t)$ cannot blow-up in a finite positive time since $\dot{x}(t)$ could become unbounded, which is impossible by (A.1).

(B) Since **X** is free from zeros the projection of its orbits in the x-axis are unbounded sets. In fact, $x(t) \to +\infty$ when $t \to +\infty$, as we show now.

Under integration of equation 3, for $t > 0$, we get

$$\dot{x} - \dot{x}_0 = \int_0^t (E_0 + p)\,\mathrm{d}t - (x - x_0),$$

and since

$$\int_0^t (E_0 + p)\,\mathrm{d}t \to +\infty$$

when $t \to +\infty$, and $\dot{x}(t)$ is bounded, we get

$$\lim_{t\to+\infty} x(t) = +\infty. \tag{9}$$

(C) We now show that for large values of t, $\dot{x}(t)$ becomes positive.

In fact, one can always assume that the initial velocity $\dot{x}_0$ is non-negative, since for $\dot{x}_0 < 0$ we get $\ddot{x} \geqslant E_0 + p(x) \geqslant e_0 > 0$ and $\dot{x} - \dot{x}_0 \geqslant e_0 t$, implying $\dot{x}(\bar{t}) > 0$ for a certain $\bar{t} > 0$.

(D) Let finally prove that

$$\lim_{t\to+\infty} \dot{x}(t) = E_0. \tag{10}$$

We assume that $\dot{x}(t)$ is not monotone when t is large, since otherwise the proof of (10) is trivial (note that $\dot{x}(t) \to L \neq E_0$ is a contradiction with equation (3)).

In fact, calling L_1, L_2 ($L_1, L_2 \in \mathbb{R}$) the higher and lower limits of $\dot{x}(t)$ for $t \to +\infty$, we get

$$L_2 = \text{higher}\lim(M_n), \tag{11}$$

(t_n, M_n) being the sequence of relative maxima of $\dot{x}(t)$.

Inserting now $t = t_n$ in equation (3) and making $t_n \to +\infty$ we get

$$0 = \ddot{x}(t_n) = E_0 - L_2, \tag{12}$$

and therefore $L_2 = E_0$.

Similarly, we get $L_1 = E_0$ by considering the sequence (t'_n, m_n) of relative minima of $\dot{x}(t)$.

Two graphics are now given for $p(x) = \frac{10x}{1+x^2}\sin(10x)$, $E_0 = 10$, see figures 1 and 2. Note the oscillating behavior in which $\dot{x}(x)$ reaches its limit $E_0 = 10$.

The reader will check that the reasoning of this section is valid for the family of differential equations

$$\left.\begin{aligned} &\ddot{x} = E_0 + p(x) - \dot{x}F(x) \\ &F > 0, \lim_{x\to+\infty} F(x) = 1 \end{aligned}\right\}, \tag{13}$$

for which $\lim_{t\to+\infty} \dot{x}(t)$ is again E_0. See figure 3, for the behavior of $\dot{x}(x)$ when in equation (13) we have

$$\left.\begin{aligned} p(x) &= \frac{\sin x}{1+x^2} \\ F(x) &= \frac{1+x^2}{2+x^2} \end{aligned}\right\}. \tag{14}$$

An open problem is the study of the limit speed for the differential equations of type

$$\ddot{x} = E_0 + p(x) - \dot{x}F(x, \dot{x}^2), \tag{15}$$

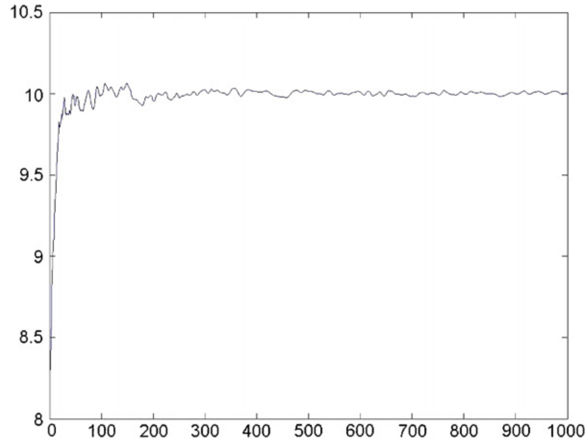

Figure 1. Initial conditions $x(0) = 0$, $\dot{x}(0) = 8$.

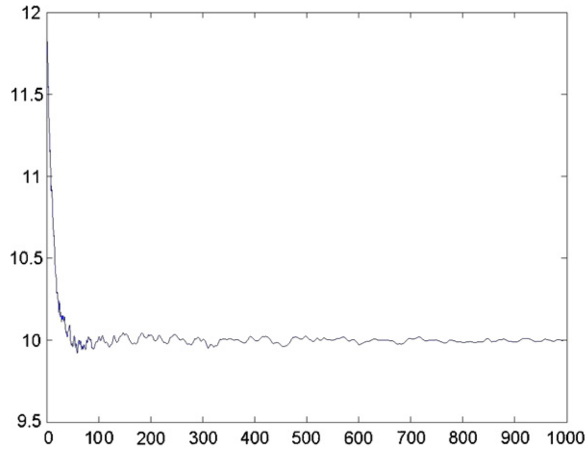

Figure 2. Initial conditions $x(0) = 0$, $\dot{x}(0) = 12$.

for convenient assumptions on F such as

$$F > 0; \qquad \lim_{x\to+\infty} F = 1,$$

and possibly others.

The case $F(x, \dot{x}^2) = G(\dot{x}^2)$ shall be studied and generalized in section 4.

3. Relativistic case in one dimension

The relativistic counterpart of equation (13) is

$$\frac{d(\gamma\dot{x})}{\mathrm{d}t} = E_0 + p(x) - \dot{x}F(x). \tag{16}$$

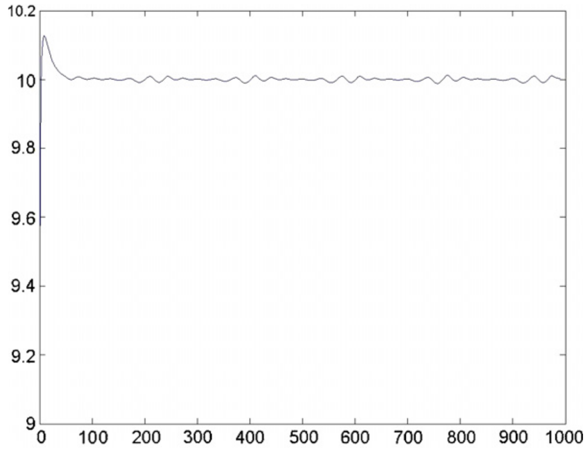

Figure 3. Initial conditions $x(0) = 0$, $\dot{x}(0) = 9$.

After easy calculations equation (16) becomes

$$\left.\begin{array}{l} \ddot{x} = (E_0 + p(x) - \dot{x}F(x))(1 - \dot{x}^2)^{3/2} \\ F > 0, \quad \lim\limits_{x\to+\infty} F(x) = 1 \\ 0 < E_0 < 1, \quad \dfrac{E_0 + p}{F} < 1 \end{array}\right\}, \tag{17}$$

where some assumptions on E_0, $p(x)$ and $F(x)$ have been added (see equations (3) and (13)). In these equations $\gamma = (1 - \dot{x}^2)^{-1/2}$ and the light speed is assumed to be equal to 1.

Let us see that $\dot{x}(t)$ has a limit E_0 when $t \to +\infty$.

In fact, the proof of section 2 holds except for the reasoning leading to equation (9). Now, since $x(t)$ is increasing for t large (see section 2.3) $\lim x(t)$ exists. Let $x(t) \to A$ ($A \in \mathbb{R}$). Then $\lim\limits_{+\infty} \dot{x}(t) = 0$ and $\lim\limits_{+\infty} \ddot{x}(t) = 0$, contradicting the limit for $t \to +\infty$ of equation (17):

$$E_0 + p(A) > 0. \tag{18}$$

Therefore, $A = +\infty$ and equations (11) and (12) are still valid for equation (17). Accordingly, we have again

$$\lim_{t\to+\infty} \dot{x}(t) = E_0. \tag{19}$$

Note that we have not included in equation (16) the radiation-reaction term, [6],

$$\left.\begin{array}{l} \dfrac{2}{3}\dfrac{1}{1 - v^2}[\gamma^2\ddot{v} + 3\gamma^4 v\dot{v}^2] \\ v = \dot{x} \end{array}\right\}, \tag{20}$$

obtained from the Lorentz–Dirac term

$$\left.\begin{array}{l} \dfrac{2}{3}\left(\dfrac{\mathrm{d}a^\mu}{\mathrm{d}\tau} - (a^\lambda a_\lambda)v^\mu\right) \\ \tau = \text{ proper time} \\ \lambda, \mu = 1, \ldots, 4 \\ v^\lambda = \text{ four velocity;} \quad a^\lambda = \text{ four acceleration} \end{array}\right\}, \tag{21}$$

when the space dimension is 1 ($q = c = 1$) for two reasons.

In fact, the basic equation

$$\ddot{x} = (1 - \dot{x}^2)^{3/2}(E_0 - \dot{x}) \tag{22}$$

including radiation-reaction terms becomes

$$\dot{v} = \left[E_0 - \dot{x} + \frac{2}{3}\frac{1}{(1-v^2)^{1/2}}\left(\ddot{v} + \frac{3v\dot{v}^2}{1-v^2}\right)\right]. \tag{23}$$

But in this equation the radiation term

$$\frac{2}{3}\frac{1}{(1-v^2)^{1/2}}\left(\ddot{v} + \frac{3v\dot{v}^2}{1-v^2}\right) \tag{24}$$

cannot be considered a perturbation of the term E_0, since the radiation term can be unbounded and all the techniques in this paper are based on perturbing functions $p(x)$ which are bounded.

The readers should also be warned that some physicists [7] are of the opinion that radiation is not an individual phenomenon of accelerated charges, and that it is only a statistical phenomenon. Stimulated radiation would arise in this way. For these physicists radiation-reaction could be absent in our equations.

4. Limit speed in $\mathbb{R}^2$: non-relativistic case

In this section, we study the limit behavior of $(\dot{x}, \dot{y})$ in the differential equations

$$\left.\begin{array}{ll} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}, & \dfrac{\mathrm{d}\dot{x}}{\mathrm{d}t} = -\dot{x}F(\dot{x}^2+\dot{y}^2) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dot{y}, & \dfrac{\mathrm{d}\dot{y}}{\mathrm{d}t} = -\dot{y}F + E_0 + p(y) \\ F, p \in C^{\omega}, & E_0 \in \mathbb{R}^+ \\ E_o + p \geqslant a > 0, & 0 < m \leqslant F \leqslant M \\ \lim\limits_{y\to+\infty} p(y) \to 0, & uF(u^2) \text{ monotonous} \end{array}\right\}, \tag{25}$$

and we prove that under the above conditions

$$\lim_{t\to+\infty}(\dot{x}, \dot{y}) = (0, L), \tag{26}$$

with L being the unique solution of the equation

$$LF(L^2) = E_0. \tag{27}$$

Equation (25) models the motion of a unit charge in a plane when a electric field $E_0 + p(y)$ parallel to the y-axis and friction $-\dot{\mathbf{x}}F(\dot{\mathbf{x}}^2)$ are present.

Note that $uF(U^2)$ is monotonous if F' is positive $\left(\frac{\mathrm{d}}{\mathrm{d}u}(uF) = F + 2u^2F'\right)$.

4.1.

We first prove that $\dot{x} \to 0$ when $t \to +\infty$.

Indeed, the solutions $x(t), y(t), \dot{x}(t), \dot{y}(t)$ of equations (25) are defined for any $t > 0$ since the blow-up time of the vector field $\mathbf{X} = (\dot{x}, \dot{y}, -\dot{x}F, -\dot{y}F + E_0 + p(y))$ is given by

$$\left.\begin{array}{l} T = \displaystyle\int_0^\infty \frac{\mathrm{d}s}{\|\mathbf{X}\|} \geqslant \int_0^\infty \frac{\mathrm{d}s_1}{\sqrt{k''\rho^2+k^2}} \geqslant \int_0^\infty \frac{\mathrm{d}\rho}{\sqrt{k''\rho^2+k^2}} = +\infty \\ k'' > 0, \quad \rho^2 = \dot{x}^2 + \dot{y}^2 \end{array}\right\}, \tag{28}$$

with $\mathrm{d}s$ and $\mathrm{d}s_1$ being the Euclidean arc elements in $\mathbb{R}^4$ and $\mathbb{R}^2$, respectively.

Therefore $x(t), y(t), \dot{x}(t), \dot{y}(t)$ are bounded for finite values of t $(t > 0)$.

Moreover, $\dot{x}(t)$ and $\dot{y}(t)$ are also bounded when $t \to +\infty$. This follows immediately from the second and fourth equations (25). Note that the local extrema of $\dot{y}$ lie on the $\mathbb{R}^3$-surface

$$-\dot{y}F + E_0 + p(y) = 0. \tag{29}$$

This surface is obviously bounded in $\dot{y}$ (note that $F \geqslant m > 0$). Note also that $\frac{\mathrm{d}\dot{y}}{\mathrm{d}t}$ is negative above this surface and positive below it.

Let us finally prove that $\dot{x}(t) \to 0$ for any value of $\dot{x}(0)$.

In fact, assume $\dot{x}(0) > 0$ (the case $\dot{x}(0) = 0$ leads to the vertical motion of section 2). Then $\dot{x}(t) > 0, \forall t > 0$ since $\dot{x}(t = \bar{t}) = 0$ $(\bar{t} \in \mathbb{R}^+)$ implies a vertical motion for $t \geqslant \bar{t}$, which is impossible by the analyticity of equations (25).

By the second of equations (25) we get $\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} < 0$. Therefore $\dot{x}(t)$ has a limit L, $0 \leqslant L \leqslant \dot{x}_0$. L must be zero since by integrating the equation $\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} < -\dot{x}F$ we get

$$L(\dot{x})|_{\dot{x}_0}^{L} = -\int_0^{+\infty} F\,\mathrm{d}t, \tag{30}$$

and since $\int_0^{+\infty} F\mathrm{d}t$ is unbounded, see equations (25), L must vanish.

4.2.

We now prove that $\dot{y} \to L$ when $t \to +\infty$, L being the solution of equation (27).

In fact, integrating the last of equations (25) we get

$$\dot{y} - \dot{y}_0 = \int_0^t (E_0 + p(y))\,\mathrm{d}t - \int_0^t F\,\mathrm{d}y, \tag{31}$$

and

$$\int_0^t (E_0 + p(y))\,\mathrm{d}t - (\dot{y} - \dot{y}_0) = \int_0^t F\,\mathrm{d}y \leqslant M(y - y_0). \tag{32}$$

Therefore $\lim_{t\to+\infty} y = +\infty$ (note that $\dot{y}$ is globally bounded). Proceeding now as in section 2.D we get equation (27).

The reader can check that the limit speed L persists when the perturbation term $p(y)$ is substituted into equations (25) by a term $p(x, y)$ which satisfies the requirements of equations (25) with the small correction:

$$\lim_{y\to+\infty} p(x, y) = 0, \tag{33}$$

as in the example

$$p = \frac{\sin x}{1 + y^2} - \frac{y}{1 + y^2}. \tag{34}$$

Further extensions of the results of this section when the vector field $(0, E_0)$ is perturbed in the form $\overline{E}_p = (p_1(x, y), p_2(x, y))$, $p_1 \neq 0$ and $\left(p_1^2 + p_2^2\right)$ small in relation with E_0, have been tried without success. This can be ascribed to the difficulty of getting now the limit of $\dot{x}(t)$ when $\to +\infty$, since we have now

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = -\dot{x}F + p_1(x, y), \tag{35}$$

and it is problematic to get $(-\dot{x}F + p_1)$ to have a constant sign opposite to $\dot{x}$.

It could also be interesting to study equations (25) when F depends not only on $\dot{\mathbf{x}}^2$ but also on $\mathbf{x}$.

The study of an isotropic friction of the form $-\overline{\overline{F}}(\dot{x}^2 + \dot{y}^2)\dot{\mathbf{x}}$, $\overline{\overline{F}}$ standing for a 2×2 matrix, can also be of interest.

5. Relativistic case in two dimensions

We now study the limit behavior of $\dot{x}(t)$, $\dot{y}(t)$ when $t \to +\infty$ for the equations

$$\left.\begin{array}{l}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\dot{\mathbf{x}}) = (0, E_0 + p(y)) - F(\dot{\mathbf{x}}^2)\dot{\mathbf{x}} \\ \mathbf{x} = (x, y) \\ E_0 > 0, \quad E_0 + p(y) \geqslant a > 0 \\ \frac{E_0 + p}{F} \leqslant \frac{1}{2}, \quad F \leqslant M \\ E_0 + p \geqslant e_0 > 0\end{array}\right\}. \tag{36}$$

Note that in this case $|\dot{x}|$ and $|\dot{y}|$ are bounded by 1, and therefore escape to infinity (that is, $x^2(t) + y^2(t) \to +\infty$ in finite time) is impossible.

We have only been able to trace back [8] concerning relativistic motion, but unfortunately only the case $F = 0$ (no friction) is studied.

After easy manipulations equations (36) become

$$\left.\begin{array}{l}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = -\dot{x}\gamma^{-1}[F\gamma^{-2} + \dot{y}(E_0 + p)] \\ \frac{\mathrm{d}\dot{y}}{\mathrm{d}t} = -\gamma^{-1}[-F\dot{y}\gamma^{-2} + (1 - \dot{y}^2)(E_0 + p)]\end{array}\right\}. \tag{37}$$

(A) Let us first prove that for

$$\frac{E_0 + p}{F} \leqslant \frac{1}{2}, \tag{38}$$

$v = (\dot{x}^2 + \dot{y}^2)^{1/2}$ cannot approach the speed of light ($v = 1$).

In fact, we get from equations (37):

$$\frac{\mathrm{d}}{\mathrm{d}t}(v^2) = 2\gamma^{-3}(-v^2F + (E_0 + p)\dot{y}), \tag{39}$$

and by assumption (38) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(v^2) < 2\gamma^{-3}(E_0 + p)(\dot{y} - 2v^2). \tag{40}$$

Consequently $\dot{v}^2$ is negative near $v = 1$ (remember that $E_0 + p$ is positive and insert $v = 1$ in the right-hand side of equation (40)). Therefore v^2 cannot approach the value $v = 1$.

(B) Let us now prove, as in section 4, that when $\dot{x}_0 > 0$, $\dot{x}(t)$ decreases with t.

In fact, from section 5.1 we can write

$$\left.\begin{array}{l}1 - v^2 \geqslant b > 0 \\ \text{that is,} \\ 1 - b \geqslant \dot{y}^2\end{array}\right\}, \tag{41}$$

and $b \geqslant b_0 > 0$, since v^2 cannot approach the value $v = 1$.

Therefore the term $F\gamma^{-2} + \dot{y}(E_0 + p)$ of the first of equations (37) is greater than

$$Fb - \sqrt{1 - b}(E_0 + p), \tag{42}$$

which is positive in the interval $b \in [b_0, 1]$ on which b varies (note that b_0 depends on the particular solution of equations (36) under hand).

Accordingly $F\gamma^{-2} + \dot{y}(E_0 + p)$ will be positive if we can choose F such that

$$Fb \geqslant \sqrt{1 - b}(E_0 + p), \tag{43}$$

that is

$$\left.\begin{array}{l} F \geqslant \dfrac{\sqrt{1-b}(E_0+p)}{b} \\ b \in [b_0, 1] \end{array}\right\}. \tag{44}$$

Note that when F is chosen in such a way that equation (44) holds, for a particular solution P of equation (36), the same thing will happen for nearby solutions if we prove that $b =$ lower bound of $v^2(t)$ is a continuous functions of the initial data $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ (see equation (41)). But this in turn implies that the bound F_0 of F (see equation (44)) defined by

$$F_0 = (E_0 + p)\mathrm{Max}\sqrt{\frac{1-b}{b^2}}, \tag{45}$$

holds for a family $\mathcal{F}$ of solutions around the particular solution P.

The globalization of $\mathcal{F}$ to the totality of all the solutions of equation (36) will be settled in a future paper.

Under these conditions the term $F\gamma^{-2} + \dot{y}(E_0 + p)$ is positive and $\frac{\mathrm{d}\dot{x}}{\mathrm{d}t}$ will be negative when $\dot{x}_0 > 0$ and $\frac{E_0+p}{F} \leqslant \frac{1}{2}$.

This implies that $L = \lim\limits_{t\to+\infty} \dot{x}(t)$ exists (and is zero), since taking limits on the first of equations (37) we get

$$0 = -L \lim_{t\to+\infty} B(t), \tag{46}$$

$B(t)$ being a positive bounded function of t (note that $L \neq \pm 1$, as was explained at the beginning of this section).

Finally, from equation (46) we get

$$L = 0. \tag{47}$$

The case $\lim\limits_{t\to+\infty} B(t) = 0$ has been excluded since (see subsection C) for large values of t the sign of $\dot{y}$ in equation (37) is positive.

(C) We show now that $\lim\limits_{t\to+\infty} y(t) = +\infty$.

In fact, from the second of equations (37) we get (note that $F \leqslant M$)

$$\dot{y} - \dot{y}_0 = -\int_0^t F\gamma^{-3}\mathrm{d}y + \int_0^t \gamma^{-1}(1-\dot{y}^2)(E_0+p)\mathrm{d}t), \tag{48}$$

and since

$$\left.\begin{array}{l} \displaystyle\int_0^t F\gamma^{-3}\mathrm{d}y \leqslant kb^{3/2}(y-y_0) \\ (1-v^2)^{1/2}(1-\dot{y}^2) \geqslant (1-v^2)^{3/2} \geqslant b^{3/2} \\ E_0 + p \geqslant a > 0 \end{array}\right\}, \tag{49}$$

we get

$$kb^{3/2}(y-y_0) \geqslant -(\dot{y}-\dot{y}_0) + b^{3/2}at, \tag{50}$$

and therefore $(y - y_0) \to +\infty$ when $t \to +\infty$.

(D) Finally, proceeding as in the above sections and taking the limit of the second of equations (37) when $t_n \to +\infty$ (at the local maxima and minima of $\dot{y}(t)$, when $\dot{y}(t)$ is not monotonous at $t \to +\infty$) we get

$$\left.\begin{array}{l} 0 = (1-L^2)^{1/2}[F(L^2)L(1-L^2) + (1-L^2)E_0] \\ L = \lim_{t\to+\infty} \dot{y}(t) \end{array}\right\}, \tag{51}$$

and since $L \neq \pm 1$, we get

$$0 = F(L^2)L - E_0, \tag{52}$$

as we desired to prove.

6. Final remarks

It would be interesting to study the limit speeds of the solutions of equations (25) when the inequalities $0 < m \leqslant F \leqslant M$ no longer hold, and therefore equation (32) is possibly false. Nevertheless the integral $\int_0^{+\infty} F \, dt$ in equation (30) is still divergent for $m = 0$, when $F \in C^\omega$ (and F does not vanish identically).

Similar observations can be pointed out in the following cases:

(i) Equation (36) when F is unbounded, in which case equation (48) can be false.
(ii) Equation (39) when $F(L) = 0$.

Note that for charges moving in $\mathbb{R}^3$ the plane π_0 defined by $\mathbf{x}_0; \dot{\mathbf{x}}_0, \mathbf{E}$ is invariant for the flow associated with the equations

$$\left.\begin{array}{l} \ddot{\mathbf{x}} = \mathbf{E} - F\dot{\mathbf{x}} \\ \mathbf{E} = \mathbf{E}_0 + \mathbf{p} \\ \mathbf{p} \parallel \mathbf{E}_0 \end{array}\right\}, \tag{53}$$

and

$$\frac{d}{dt}(\gamma \mathbf{x}) = \mathbf{E} - F\dot{\mathbf{x}}. \tag{54}$$

Therefore the changes move in $\mathbb{R}^3$ on these planes π_0. Accordingly, they behave as if they were moving in $\mathbb{R}^2$.

The study of equations (25) when friction is anisotropic (that is, when it is of the form $-\overline{\overline{F}}(\mathbf{x}, \dot{\mathbf{x}})\dot{\mathbf{x}}$, and $-\overline{\overline{F}}$ is invariant under the transformation $\mathbf{x} \to \mathbf{x}, \dot{\mathbf{x}} \to -\dot{\mathbf{x}}$, $\overline{\overline{F}}$ standing for a 2×2 matrix whose elements depends on $(\mathbf{x}, \dot{\mathbf{x}})$, $\overline{\overline{F}} \neq \lambda(\mathbf{x}, \dot{\mathbf{x}}) I_2$, $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, is of physical interest and shall be studied in a next paper.

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